The random walk with stochastic sojourn times

van der Genugten, Berend Bernhard

Publication date:
1969

CHAPTER I

INTRODUCTION

§ 1. SUMMARY.

In this thesis we will investigate random walks which, upon reaching a state x, stay in x during a certain time (sojourn time). The distribution of this time only depends on x and not on the previous development of the process. So an essential feature of random walks, viz. the independence of the jumps, is maintained. Therefore we may call such a process *a random walk with stochastic sojourn times (s.r.w.)*. More general, if the starting point is also stochastic we will call such a process *a generalized random walk with stochastic sojourn times (s.g.r.w.)*. Note that the s.g.r.w. is a semi-Markov process.

In our investigation of the s.g.r.w. we will of course only be interested in the properties in which the s.g.r.w. differs from the random walk. In chapter II attention is paid to the existence of a probability space for the s.g.r.w. Also a zero-one law for the s.g.r.w. is derived. In chapter III criteria are given for honesty and dishonesty. The chapters IV-VI are preparatory ones. Here we study integrals of non-negative functions with respect to transient renewal measures. Although not needed in the sequel, asymptotic and exact expressions for renewal measures are derived as well. In chapter VII the expectation of the time needed for infinitely many jumps for the s.r.w. is studied, and also the expectation of the ladder epochs. The results follow from the chapters IV-VI. The chapters VII and IX contain respectively weak and strong laws of large numbers for the s.g.r.w. In chapter X the existence and the form of limiting distributions are investigated. Also a generalisation of a theorem originated with Harris and Robbins is given. Chapter XI deals with a system of s.g.r.w.'s with Poisson-input. Finally, an example of a s.g.r.w. occurring in the theory of neutron diffusion is given.

It was the example of chapter XI that led us to the study of the s.g.r.w. However, the purpose of our investigation of the s.g.r.w. is not its applications, but is more theoratical, viz. to compare the behaviour of the s.g.r.w. with that of the common random walk, at the same time extending the theory in those fields of the theory of probability where this was necessary.

For previous results, related to and part of this thesis, we refer to [26] - [28] of the references.

§ 2. NOTATIONS AND CONVENTIONS.

With R, R_k or R^k for integer k we denote the set $(-\infty, \infty)$, and with $\mathcal{B}$, $\mathcal{B}_k$ or $\mathcal{B}^k$ the Borel sets of R, R_k and R^k. Let R_∞ denote the product space of R_1, $R_2, \ldots$, and $\mathcal{B}^\infty$ the Borel sets of R^∞. Similarly we define R^∞ and $\mathcal{B}^\infty$. Finite product spaces are indicated e.g. by

$$\left(\mathop{\times}_{k=0}^{n} R_k, \mathop{\times}_{k=0}^{n} \mathcal{B}_k \right).$$

Set $R^+ = [0, \infty)$ and $R^- = (-\infty, 0]$. The set $A + x$, for $x \in R$ and $A \subset R$, denotes the *translation* of A to the right over a distance x. With I_A or $I\{A\}$ the *indicator function* of the set A is denoted.

The *variation* of a signed measure G is indicated by $|G|$. Note that $|G|$ is a measure. The *total variation* of G is denoted by $\|G\|$. We call G finite if $\|G\| < \infty$.

We say that the integral of a (complex) Borel function g with respect to the signed measure G on $(R, \mathcal{B})$ exists or is *convergent* if $\int |g(x)| \; |G|\{dx\} < \infty$. Otherwise it is called *divergent*. If g is non-negative, if G is a measure and if the integral of g with respect to G is divergent we write $\int g(x)G\{dx\} = \infty$.

Convolutions are written as products and powers. If confusion can arise with common products and powers we will use the convolution symbol $*$.

A *distribution* is a probability measure on $(R, \mathcal{B})$. The *distribution function* of a distribution F is indicated with the same symbol and defined by $F(x) = F\{(-\infty, x)\}$, $x \in R$.

The distribution F is *lattice* with span λ, or briefly, has span λ, if F is concentrated on $\{n\lambda : n \text{ integer}\}$ and not on $\{n\lambda' : n \text{ integer}\}$ for any $\lambda' > \lambda$. If there does not exist such λ then F is called *non-lattice*.

The distribution F is called *singular* if F is restricted to a set of Lebesgue measure zero, and *non-singular* otherwise.

The distribution of a random variable $\underline{x}$ is *proper* if $P\{|\underline{x}| < \infty\} = 1$ and *defective* otherwise.

For the first moment μ of a distribution F we will not follow the conventions on the existence of integrals as given before.

We say

that $\mu = \int x\, F\{dx\}$ is finite if $\int |x|\, F\{dx\} < \infty$,

that $\mu = \infty$ ($\mu = -\infty$) if $\int_{R^+} x F\{dx\} = \infty$ ($< \infty$)

and $\int_{R^-} |x|\, F\{dx\} < \infty$ ($= \infty$),

and that μ does not exist if $\int_{R^+} x F\{dx\} = \int_{R^-} |x| F\{dx\} = \infty$.

The k^{th} derivative of a function g is denoted by $g^{(k)}$, $k = 0, 1, \ldots$, where $g^{(0)} = g$. In particular, we set $g' = g^{(1)}$.

The abbreviations i.o., a.s. and a.e. stand for, respectively, infinitely often, almost sure, and almost everywhere with respect to Lebesgue measure.

Finally, $\langle\langle x \rangle\rangle$ stands for max $\{n : n \leqslant x,\ n \text{ integer}\}$, $\langle x \rangle$ for min $\{n : n \geqslant x,\ n \text{ integer}\}$. The symbol $<<$ denotes *absolute continuity*.

§ 3. THE RANDOM WALK AND ITS RENEWAL MEASURE.

In this section we mention those definitions and elementary properties of random walks which we need and to which we will not refer explicitly in the next chapters (see Feller [1] VI. 10, XI. 9, XII. 1, 2 and XVIII. 8, examples).

Let $\underline{y}_1, \underline{y}_2, \ldots$ be independent identically distributed random variables and let

$$\underline{S}_0 = 0; \underline{S}_n = \underline{y}_1 + \ldots + \underline{y}_n,\ n = 1, 2, \ldots$$

Then the process $\{\underline{S}_n, n = 0, 1, \ldots\}$ is called a *random walk (r. w.)*.

The random variable $\underline{S}_n$ is interpreted as the position at time (epoch) n of a particle moving with random jumps, y_n being the *jump* at time n. For a set $A \subset R$ the event $\{\underline{S}_n \epsilon A\}$ is called a *visit* to A at epoch n.

Let F be some distribution. Then the measure U, defined by

$$U = \sum_{k=0}^{\infty} F^k ,$$

is called the *renewal measure* belonging to F.

With every r.w. $\{\underline{S}_n\}$ there goes a distribution F and conversely. Therefore probabilistic definitions and properties of $\{\underline{S}_n\}$ can be formulated in terms of $\{\underline{S}_n\}$ as well as F. We can, e.g., speak of $\{\underline{S}_n\}$ being lattice, or of the renewal measure U belonging to $\{\underline{S}_n\}$. Note that $U\{A\}$ equals the expectation of the number of visits of $\{\underline{S}_n\}$ to the set A.

From now on F will stand for the distribution of the r.w. $\{\underline{S}_n\}$, and it is assumed that *F is not degenerate at* 0. Moreover, as far as defined, μ and μ_2 stand for the first and second moment of F.

We call F or $\{\underline{S}_n\}$ *transient* if $U\{I\} < \infty$ for all finite intervals I, and *recurrent* otherwise.

Let F be recurrent. If F is non-lattice every interval I is visited infinitely often a.s., and $U\{I\} = \infty$. If F is lattice with span λ then every point $\{n\lambda\}$, integer n, is visited infinitely often a.s., and $U\{n\lambda\} = \infty$. Moreover, if μ exists then F is recurrent if and only if $\mu = 0$.

Let F be transient. Then every finite interval I is visited only finitely often a.s. For any interval I of finite length $h > 0$ we have

$$U\{I\} \leqslant U\ \{[-h,\ h]\}$$

Suppose $|\mu| = \infty$ or μ does not exist. Then $U\{I+x\} \longrightarrow 0$, $|x| \longrightarrow \infty$, for every finite interval I.

Suppose $0 < \mu < \infty$. Then we have: if F is non-lattice then $U\{I+x\} \longrightarrow \mu^{-1}\ |I|$ and $U\{I-x\} \longrightarrow 0$, $x \longrightarrow \infty$, for every interval I of finite length $|I|$, and if F is lattice with span λ then $U\{n\lambda\} \longrightarrow \mu^{-1}\lambda$ and $U\{-n\lambda\} \longrightarrow 0$, $n \longrightarrow \infty$

We call F or $\{\underline{S}_n\}$ *ascending* if $\underline{S}_n \longrightarrow \infty$ a.s., $n \longrightarrow \infty$, *descending* if $\underline{S}_n \longrightarrow -\infty$ a.s., $n \longrightarrow \infty$, and *oscillating* otherwise.

In the latter case we have
$\liminf_{n\to\infty} \underline{S}_n = -\infty$ and $\limsup_{n\to\infty} \underline{S}_n = \infty$, a.s. $n \longrightarrow \infty$.

If μ exists then $\{\underline{S}_n\}$ is oscillating for $\mu = 0$, ascending for $0 < \mu \leqslant \infty$ and descending for $-\infty \leqslant \mu < 0$. So in this case it is clear that the notions oscillating and recurrent are identical and that the same holds for transient on the one side and for ascending and descending on the other side.

If μ does not exist then every ascending and descending $\{\underline{S}_n\}$ is obviously transient. However, oscillating $\{\underline{S}_n\}$ can be recurrent as well as transient.

For fixed n let $\underline{y}_1^* = \underline{y}_n, \ldots, \underline{y}_n^* = \underline{y}_1$, and $\underline{S}_0^* = 0$, $\underline{S}_k^* = \underline{y}_1^* + \ldots + \underline{y}_k^*$ for $k = 1, \ldots n$. If A is any event defined by $(\underline{S}_0, \ldots, \underline{S}_n)$ and A^* the event defined in the same way by $(\underline{S}_0^*, \ldots, \underline{S}_n^*)$ then A and A^* are called *dual*.

Since $(\underline{S}_0, \ldots, \underline{S}_n)$ and $(\underline{S}_0^*, \ldots, \underline{S}_n^*)$ have the same joint distribution dual events have the same probability.

The r^{th} point of the sequence $\{(n, \underline{S}_n), n = 1, 2, \ldots\}$ for which

$$\underline{S}_n > \underline{S}_0, \ \ldots, \ \underline{S}_n > \underline{S}_{n-1}$$

is called the r^{th} *strict ascending ladder point* of $\{\underline{S}_n\}$. Note that this point does not always exist a.s. Since the r.w. $\{\underline{S}_n\}$ starts anew at any epoch the r^{th} strict ascending ladder point can be written as $(\underline{t}_1^+ + \ldots + \underline{t}_r^+, \underline{h}_1^+ + \ldots + \underline{h}_r^+)$ where the pairs $(\underline{t}_k^+, \underline{h}_k^+)$ are independent and identically distributed. The first element of the ladder point is called the *ladder epoch* and the second one the *ladder height*. Set $\underline{t}^+ = \underline{t}_1^+$, the first strict ascending ladder epoch and $\underline{h}^+ = \underline{h}_1^+$, the first strict ascending ladder height. Note that the distribution of $\underline{t}^+$ and the distribution H^+ of $\underline{h}^+$ are possibly defective.

The *weak ascending ladder points* are defined similarly with $>$ replaced by $\geqslant$. Let $\underline{\tilde{t}}^+$ stand for the first weak ascending ladder epoch and $\underline{\tilde{h}}^+$, with distribution $\tilde{H}^+$, for the first weak ascending ladder height. The connection between H^+ and $\tilde{H}^+$ is simply:

$$\tilde{H}^+ = \gamma\, \psi^0 + (1 - \gamma)\, H^+,$$

where ψ^0 is the distribution degenerate at 0, and γ the probability that $\{\underline{S}_n\}$ returns to 0 without a prior visit to

$(0,\infty)$. Note that $0 \leqslant \gamma < 1$ since F is not degenerate at 0. Let ψ^+ and $\tilde{\psi}^+$ be the renewal measures belonging to H^+ and $\tilde{H}^+$. Then we have:

$$\psi^+ = (1 - \gamma)\tilde{\psi}^+.$$

The *strict and weak descending ladder points* are defined by changing $>$ or $\geqslant$ into $<$ and $\leqslant$. We follow the same notations. However, the superscript plus is replaced by minus. Using the duality principle it easily follows that γ is also the probability that $\{\underline{S}_n\}$ returns to 0 without a prior visit to $(-\infty, 0)$ and therefore

$$\tilde{H}^- = \gamma\, \psi^o + (1-\gamma)\, H^-$$

$$\psi^- = (1 - \gamma)\tilde{\psi}^-.$$

The r.w. $\{\underline{S}_n\}$ is ascending if and only if H^+ is proper and H^- defective. The same holds for the distributions of $\underline{t}^+$ and $\underline{t}^-$ and then $E\{\tilde{\underline{t}}^+\} = (1-\gamma)\ E\{\underline{t}^+\} < \infty$. A similar result holds for $\{\underline{S}_n\}$ descending.

The r.w. $\{\underline{S}_n\}$ is oscillating if and only if both H^+ and H^- are proper. The same holds for the distributions of $\underline{t}^+$ and $\underline{t}^-$ and then $E\{\underline{t}^+\} = E\{\underline{t}^-\} = \infty$.

The process $\{\underline{S}_n\}$ has the fixed initial state 0. More general the initial state can be given by the random variable $\underline{y}_0$ with distribution F_0, independent of $\underline{y}_1, \underline{y}_2, \ldots$. If $\underline{Z}_n = \underline{y}_0 + \underline{S}_n$, we call the process $\{\underline{Z}_n, n=0,1,\ldots\}$ *a generalized random walk* (*g.r.w.*).

Let $U_0 = UF_0$, then $U_0\{I\}$ equals the expectation of the number of visits of $\{\underline{Z}_n\}$ to I.

We call $\{\underline{Z}_n\}$ non-lattice if $\{\underline{S}_n\}$ is so. In particular, if F_0 is absolutely continuous or if some F^m is non-singular we call $\{\underline{Z}_n\}$ non-singular. Otherwise we call $\{\underline{Z}_n\}$ singular.

We call $\{\underline{Z}_n\}$ lattice with span λ if $\{\underline{S}_n\}$ is so and if F_0 is restricted to $\{n\lambda\}$, integer n.

Note that both cases are not exhausting.